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LETTER TO THE EDITOR

Integrability of a one-dimensional Fermi system with magnetic impurities

E H Rezayi† and J Sak‡

† Center for Pure and Applied Physical Sciences, University of California, San Diego, La Jolla, CA 92093, USA

‡ Serin Physics Laboratory, Rutgers University, New Brunswick, NJ 08903, USA

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Abstract. A one-dimensional metal containing interacting electrons moving in both directions and magnetic impurities is studied. The model is shown to be integrable by the Bethe ansatz provided that an algebraic relation is satisfied between the coupling constants of the electron–electron and electron–impurity interactions.

In the past few years the method of Bethe (Bethe 1931, Gaudin 1967, Yang 1967) known as the Bethe ansatz has been employed to find the spectrum of some field theory models in $1+1$ dimensions. Two recent examples are the chiral invariant Gross–Neveu (CIGN) model (Belavin 1979, Andrei and Lowenstein 1980) and the Kondo problem (Andrei 1980, Vigman 1980). The former is the field theoretic analogue of the one-dimensional Fermi system with backscattering but without *Umklapps*. This model is of physical interest for quasi-one-dimensional conductors (Solyom 1979 and references therein). In this paper we consider a more general model that contains both CIGN and Kondo models as special cases. We consider the Hamiltonian

$$\begin{aligned} H = &-\mathrm{i}\int \psi^{+}_{1\alpha}\frac{\partial}{\partial x}\psi_{1\alpha}\,\mathrm{d}x + \mathrm{i}\int \psi^{+}_{2\alpha}\frac{\partial}{\partial x}\psi_{2\alpha}\,\mathrm{d}x \\ &+ J\int \psi^{+}_{0\alpha}\boldsymbol{\sigma}_{\alpha\beta}\psi_{0\beta}\cdot(\psi^{+}_{1\gamma}\boldsymbol{\sigma}_{\gamma\delta}\psi_{1\delta}+\psi^{+}_{2\gamma}\boldsymbol{\sigma}_{\gamma\delta}\psi_{2\delta})\,\mathrm{d}x \\ &+ U\int \psi^{+}_{1\alpha}\boldsymbol{\sigma}_{\alpha\beta}\psi_{1\beta}\cdot\psi^{+}_{2\gamma}\boldsymbol{\sigma}_{\gamma\delta}\psi_{2\delta}\,\mathrm{d}x + V\int \psi^{+}_{0\alpha}\psi_{0\alpha}(\psi^{+}_{1\beta}\psi_{1\beta}+\psi^{+}_{2\beta}\psi_{2\beta})\,\mathrm{d}x \\ &+ \Phi\int \psi^{+}_{1\alpha}\psi_{1\alpha}\psi^{+}_{2\beta}\psi_{2\beta}\,\mathrm{d}x. \end{aligned} \tag{1}$$

ψ_1 and ψ_2 are the fields of the right- and left-going electrons, ψ_0, with vanishing bandwidth, corresponds to the impurities; the terms with the coupling constant J are the Kondo interaction, U is the spin exchange between electrons which is equivalent (leaving aside the problems with the cut-offs) to the backscattering; the last two terms, V and Φ, are potential scatterings between electrons and impurities and between left- and right-going electrons.

In addition to having a direct relevance to quasi-one-dimensional conductors with magnetic impurities, our model provides a testing ground with regard to questions

such as how the presence of the left-going electrons affects the magnetic behaviour of impurities. For example, the nature of the conduction electron induced RKKY-type interaction among impurities is of some interest. No such interaction exists in the Kondo model (Andrei 1980, Vigman 1980). Despite the fact that the Kondo model with only a single impurity is physically significant, the question of the existence of RKKY interactions in integrable models remains an interesting one. In this paper we present the conditions for the integrability of the model and obtain the equations that determine the spectrum, while postponing discussions of more detailed questions to forthcoming publications.

It turns out that the qualitative features of the problem, such as integrability, do not depend on the values of the potential terms V and Φ; their presence only makes the formulae more cumbersome. For this reason we perform the subsequent analysis only for the special case $V=0$ and $\Phi=0$.

The numbers of left- and right-going electrons are separately conserved as well as the number of impurities. This enables us to transcribe the problem in the first quantisation. Moreover, it is advantageous to consider the three kinds of particles as identical but equipped with a new quantum number which is sometimes called 'purity' and which distinguishes between different kinds of particles. If the ith particle is a right-going electron, its purity $b_i=1$, if it is a left-going electron, $b_i=-1$ and, finally, if it is an impurity, $b_i=0$. Now we can rewrite the Hamiltonian (1), having set $V=\Phi=0$, in the form

$$H=-\mathrm{i}\sum_{j=1}^{N} b_j\frac{\partial}{\partial x_j}+J\sum_{i<j}\delta(x_i-x_j)\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j(b_i^2-b_j^2)^2$$
$$-\tfrac{1}{2}U\sum_{i<j}\delta(x_i-x_j)\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j b_i b_j(1-b_i b_j). \tag{2}$$

The wavefunctions depend on space, spin and purity coordinates: $\Psi(x_1,\ldots,x_N;\ \sigma_1,\ldots,\sigma_N;\ b_1,\ldots,b_N)$. We do not lose any eigenstates if we require that the wavefunctions be antisymmetric with respect to the exchange of the coordinates of two particles. This allows us to eliminate the spin operators from the Hamiltonian, using the identity

$$\delta(x_i-x_j)\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j=-\delta(x_i-x_j)-2\delta(x_i-x_j)P_k^{ij} \tag{3}$$

where P_k^{ij} is the operator which interchanges the purity coordinates of the ith and jth particle. The final form of the Hamiltonian is

$$H=-\mathrm{i}\sum_{j=1}^{N} b_j\frac{\partial}{\partial x_j}-2J\sum_{i<j}(b_i^2-b_j^2)^2\delta(x_i-x_j)P_k^{ij}-J\sum_{i<j}(b_i^2-b_j^2)^2\delta(x_i-x_j)$$
$$+U\sum_{i<j}b_i b_j(1-b_i b_j)\delta(x_i-x_j)P_k^{ij}+\tfrac{1}{2}U\sum_{i<j}b_i b_j(1-b_i b_j)\delta(x_i-x_j). \tag{4}$$

The operator (4) does not contain spin. We look for its eigenfunctions in the form given by the Bethe ansatz. Let Q be a permutation of N objects (N is the total number of electrons and impurities). The sector Q in the coordinate space is defined by the inequalities $0<x_{Q1}<x_{Q2}\ldots<x_{QN}<L$. In this sector the wavefunction has the form

$$\Psi_Q=\sum_P (Q|\xi|P)\exp\Big(\mathrm{i}\sum_{j=1}^{N}k_{Pj}\cdot x_{Qj}\Big)\prod_{l=1}^{N}\delta_{a_{Pl}b_{Ql}}. \tag{5}$$

In (5) the coefficients $(Q|\xi|P)$ and the parameters k_i are to be determined, the values of a_i's can be prescribed, e.g., by $a_1 = \ldots = a_{N_0} = 0$, $a_{N_0+1} = \ldots = a_{N_0+N_1} = 1$, $a_{N_0+N_1+1} = \ldots = a_N = -1$. The wavefunction Ψ, defined to be equal to Ψ_Q in the sector Q for all Q, must obey three conditions: (1) it must satisfy the Schrödinger equation (this condition leads to a prescription to calculate the quantities $(Q|\xi|P)$); (2) it must satisfy the periodic boundary conditions, $\Psi(\ldots x_i + L \ldots) = \Psi(\ldots x_i \ldots)$ (this will, as for free particles, determine the possible values of k_i's); (3) Ψ must belong to the representation $[2^M 1^{N-2M}]$ of the permutation group S_N. The number M is connected to the spin S of the system by the relation $S = N/2 - M$. (Since the Hamiltonian does not contain the spins, it is through this condition that the wavefunction depends on the total spin of the system.) The energy eigenvalue corresponding to the eigenfunction (5) is

$$E = \sum_{j=1}^{N} a_j k_j. \tag{6}$$

We start with the first condition which relates the jumps of the wavefunction on the boundaries of the sectors to the intensity of the delta-function interaction. The result is a set of passage formulae for computation of the coefficients $(Q|\xi|P)$. Let a, i, j be integers, $a < N$, $i, j \leq N$. Then the passage formulae can be written as

$$(Q|\xi|P(m, m+1)) = Y^{m,m+1}_{P_{m,P(m+1)}} (Q|\xi|P) \tag{7}$$

where the Yang operator $Y^{m,m+1}_{ij}$ is given by

$$Y^{m,m+1}_{ij} = \frac{4\mathrm{i}(a_i - a_j)f + [1 - 3(a_i - a_j)^2 f^2] P^{m,m+1}}{1 + 3(a_i - a_j)^2 f^2 - 2\mathrm{i}(a_i - a_j)f} \equiv a_{ij} + b_{ij} P^{m,m+1} \tag{8}$$

if $a_i \neq a_j$, and $Y^{m,m+1}_{ij}$ = identity if $a_i = a_j$. In (8) $f = J/2$ when a_i and a_j correspond to an electron–impurity pair ($a_i = 0$, $a_j = 1$, etc), and $f = U/8$ if the pair is electron–electron (e.g. $a_i = 1$, $a_j = -1$). The operator $P^{m,m+1}$ acts in the $N!$-dimensional space of the coefficients $(Q|\xi|P)$:

$$P^{m,m+1}(Q|\xi|P) = (Q(m, m+1)|\xi|P). \tag{9}$$

The passage formulae (7) overdetermine the coefficients $(Q|\xi|P)$ and there arises the problem of consistency. The operators Y must satisfy the following consistency relations (Yang 1967, Baxter 1968):

$$Y^{m,m+1}_{ij} Y^{m,m+1}_{ji} = I, \tag{10}$$

$$Y^{m,m+1}_{jk} Y^{m+1,m+2}_{ik} Y^{m,m+1}_{ij} = Y^{m+1,m+2}_{ij} Y^{m,m+1}_{ik} Y^{m+1,m+2}_{jk}. \tag{11}$$

Condition (10) is satisfied by (8) for any f and so is condition (11) when at least two of the quantum numbers a_i, a_j, a_k are equal. However, when they are all different, equation (11) is not satisfied automatically. All six possibilities lead to the same constraint on the coupling constants J and U: the problem is integrable by Bethe ansatz if

$$(4 - 3J^2)/J = (16 - 3U^2)/4U. \tag{12}$$

The existence of this integrability condition is the main new result of this paper. (In the more general case when the potential scattering interactions V and Φ do not vanish, equation (12) will be replaced by a more complicated algebraic relation depending on V and Φ.)

Assuming (12) satisfied, the calculation proceeds as usual (Yang 1967). The symmetry condition is imposed by requiring that the operators P^{ab} belong to the $[2^M 1^{N-2M}]$ representation of the symmetry group S_N. The periodic boundary conditions determine the momenta k_i through the set of N eigenvalue problems

$$X'_{j+1,j} \ldots X'_{Nj} X'_{1j} \ldots X'_{j-1,j} \cdot \Phi = \exp(ik_j L)\Phi, \qquad j = 1, 2, \ldots, N. \tag{13}$$

Here the operators X'_{ij} are defined as

$$X'_{ij} = b_{ij} - a_{ij} P^{ij}, \tag{14}$$

and the effective coupling constant c is

$$c = 2J/(1 - \tfrac{3}{4}J^2) = 2U/(1 - \tfrac{3}{16}U^2). \tag{15}$$

The two different expressions for c in (15) arise depending on the pair of values a_i, a_j; they are equal thanks to the integrability condition (12) which can be also expressed by saying that the effective coupling constants for electron–impurity and electron–electron scattering must be equal.

To write down the solution of the eigenvalue problems (13) one introduces (Yang 1967) a set of auxiliary quantities ('spin momenta') $\Lambda_1 \ldots \Lambda_M$ which are determined from the M equations

$$\left(\frac{1-\Lambda_\alpha + ic/2}{1-\Lambda_\alpha - ic/2}\right)^{N_1} \left(\frac{1+\Lambda_\alpha - ic/2}{1+\Lambda_\alpha + ic/2}\right)^{N_{-1}} \left(\frac{\Lambda_\alpha - ic/2}{\Lambda_\alpha + ic/2}\right)^{N_0} = -\prod_{\beta=1}^{M} \frac{\Lambda_\beta - \Lambda_\alpha + ic}{\Lambda_\beta - \Lambda_\alpha - ic}. \tag{16}$$

In terms of Λ_α's the 'momenta' k_j are given by

$$\exp(ik_j L) = \prod_{i \neq j} e_{ij} \prod_{\alpha=1}^{M} \frac{ia_j - i\Lambda_\alpha + c/2}{ia_j - i\Lambda_\alpha - c/2} \tag{17}$$

where e_{ij} is a number of modulus unity. A simple calculation gives $e_{ij} = \exp[-i(a_i - a_j)\phi]$, $\tan(\phi/2) = J/2$ for an electron–impurity pair ($a_j - a_i = \pm 1$) and $e_{ij} = \exp[-i(a_i - a_j)\psi]$, $\tan\psi = U/4$ for an electron–electron pair. When $a_i = a_j$, $e_{ij} = -1$.

To find the possible values of the 'momenta' k_j we take the logarithm of (17). For the electrons, $a_j = \pm 1$, we get

$$k_j = \frac{1}{L}\left(2\pi n_j + a_j N_0 \phi + 2a_j N_{-a_j}\psi + a_j(N_{a_j} - 1)\pi + \sum_{\alpha=1}^{M} [\theta(2\Lambda_\alpha - 2a_j) - a_j \pi]\right) \tag{18}$$

where $\theta(x) = -2\tan^{-1}(x/c)$, $\theta \in (-\pi, \pi)$.

The numbers n_j are arbitrary integers. To obtain the energy, we substitute (18) into (6):

$$E = \frac{1}{L}\left(2\pi \sum_{j=1}^{N} a_j n_j + N_1 \sum_{\alpha=1}^{M} \theta(2\Lambda_\alpha - 2) - N_{-1} \sum_{\alpha=1}^{M} \theta(2\Lambda_\alpha + 2) - (N_1 + N_{-1})M\pi\right) + \text{constant} \tag{19}$$

where the constant does not depend on the spin. At this point the definition of the system must be supplemented by providing some cut-off. It would have been more satisfactory to build the cut-off into the Hamiltonian but that would have rendered the model insoluble by Bethe ansatz. Therefore we follow the current practice (and

there is strong evidence (Andrei and Lowenstein 1980) that there is nothing wrong with it) to cut off the integers n_j from below by K.

Now we take the logarithm of (16) and obtain

$$N_1\theta(2\Lambda_\alpha-2)+N_{-1}\theta(2\Lambda_\alpha+2)+N_0\theta(2\Lambda_\alpha)=-2\pi J_\alpha+\sum_{\beta=1}^{M}\theta(\Lambda_\alpha-\Lambda_\beta) \tag{20}$$

where J_α are integers.

In what follows, we assume that $N_1=N_{-1}\equiv N_e/2=KL/4\pi$ are large numbers (thermodynamic limit). We also assume that the ground state is a singlet, so that $M=\frac{1}{2}N$. Let us introduce the density ρ of the numbers Λ_α as

$$N\rho(\Lambda_\alpha)=(\Lambda_{\alpha+1}-\Lambda_\alpha)^{-1} \tag{21}$$

and consider the case of antiferromagnetic coupling, $c>0$ (it is questionable that one can make any sense at all of this model in the case of ferromagnetic coupling, since, in this case, the dynamical scale (Kondo temperature) is larger than the cut-off).

In the thermodynamic limit the formula for the spin dependent part of the energy becomes

$$E=\frac{NN_e}{L}\int_{|\lambda|>B}\mathrm{d}\lambda\,\rho(\lambda)\left(\tan^{-1}\frac{2\lambda+2}{c}-\tan^{-1}\frac{2\lambda-2}{c}\right)-\frac{N_eM\pi}{L}. \tag{22}$$

The fact that the number of Λ's is M is expressed as

$$S=\tfrac{1}{2}N\int_{-B}^{B}\rho(\lambda)\,\mathrm{d}\lambda. \tag{23}$$

The integration boundary B in (23) and (24) can be eliminated and one obtains the desired dependence $E(S)$ of the energy on the magnetisation S.

Finally we have to rewrite (20) in the thermodynamic limit to obtain an equation for the density ρ

$$2\pi\rho(\lambda)=\frac{N_e}{N}\left(\frac{2c}{c^2+4(\lambda+1)^2}+\frac{2c}{c^2+4(\lambda-1)^2}\right)+\frac{2N_0}{N}\frac{2c}{c^2+4\lambda^2}$$
$$-\int_{|\lambda'|>B}\frac{2c\rho(\lambda')\,\mathrm{d}\lambda'}{c^2+(\lambda-\lambda')^2}. \tag{24}$$

The system of equations (23)–(24), when solved, allows one to find the dependence of E on S. Some preliminary results which can be obtained by solving (24) have already been announced (Rezayi and Sak 1982).

To summarise our results, we have found the condition of integrability of the model given by the Hamiltonian (1) and reduced the problem to an integral equation.

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References

Andrei N 1980 *Phys. Rev. Lett.* **45** 379
Andrei N and Lowenstein J H 1980 *Phys. Rev. Lett.* **43** 1698
Baxter R J 1968 *Ann. Phys., NY* **70** 193
Belavin A A 1979 *Phys. Lett.* **87B** 117
Bethe A 1931 *Z. Phys.* **71** 205
Gaudin M 1967 *Phys. Lett.* **24A** 55
Japardize G I and Nersesyan A A 1981 *Phys. Lett.* **85A** 23
Mattis D and Lieb E 1962 *Phys. Rev.* **125** 164
Rezayi E H and Sak J 1982 *Phys. Lett.* **89A** 451
Solyom J 1979 *Adv. Phys.* **28** 201
Vigman P B 1980 *Pisma Zh. Eksp. Teor. Fiz.* **31** 392 (Engl. transl. 1980 *JETP Lett.* **31** 364)
Yang C N 1967 *Phys. Rev. Lett.* **19** 1312